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GEODETIC ASPECTS IN THE ASSESSMENT OF THE LEAST SQUARES METHOD

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Abstract

Contemporary assessment is made of the method of least squares (MLS). For this purpose geodetic studies related to the LMS topic are summarized. Conclusions are drawn for the extended general case of adjustment and specializations. The limits are presented for the model (MLS) validity, the comparison of Lagrangean functions in geodesy and mechanics and the use of geodetic solutions in mechanics. Evaluation of the correctness of the model for geodetic network adjustment and a relevant solution for its development and application in the study of deformations are presented.

Key words: method of least squares, Lagrangean functions in geodesy and mechanics, model for geodetic network adjustment, deformations study

1. General. The method of least squares (MLS) was developed independently by Gauss (Germany) and Legendre (France) [1]. Gauss discovered it in 1794–1795 and developed it further in 1821–1823, but it was first published in 1806 by Legendre. In both cases MLS development is associated with the more accurate determination of planet orbits. However, it is widely applied in various fields of science and practice – mathematics, physics, mechanics, etc., for solving overdetermined systems of algebraic equations, i.e. when the number of equations exceeds the number of unknowns, when deriving correlation dependencies, etc. It is also a basic and broadly implemented method in geodesy for processing and evaluating the accuracy of geodetic measurements, networks, etc. At the same time a huge number of publications, including capital works, are dedicated to the development of its theory, models, algorithms and software realization for the purposes of geodesy. Suffice it to mention those of Markov [2], Helmert [3], Wolf [4,5], Chebotarev [6], Hristov [7] and others.

A great deal of the works and research of the author and associates under his guidance for the purposes of geodesy are also associated with this method and its use. Systematization and summary of this research is done here, with conclusions for the general case and specializations, evaluation of correctness and conclusions for MLS applicability, as well as its improvement in the area of geodesy and mechanics.

2. Extension of the general case of adjustment by MLS and specializations. ***2.1. Mathematical model of measurements.*** The mathematical model to determine the assessment of the sought quantities, their functions and their accuracy by the Method of Least Squares in geodesy is based on measurements free of gross and systematic errors that are normally distributed.

Relationships exist between the measured and sought, and often also given, quantities that have a stochastic and functional character. The objective of the mathematical model for processing of the geodetic measurements is to cover and approximate these relationships as precisely as possible.

Two or three kinds of quantities are possible for each problem of adjustment in geodesy by the method of least squares:

- n is the number of the observed quantities with true values L_i, $i = 1, \ldots, n$;
- m is the number of the sought quantities;
- p is the number of the initial quantities.

In some cases the sought quantities coincide with the measured ones. Between these three kinds of quantities r relationships of the type

$$f_r(X_{01}, X_{02}, \ldots, X_{0m}, L_1, L_2, \ldots, L_n, d_1, d_2, \ldots, d_p) = 0 \tag{1}$$

are possible.

If (i) the measured values with added relevant corrections are introduced in (8) instead of the true values, (ii) if the relevant assessments are placed for the sought parameters, and (iii) if the given quantities are constant values, the following will be obtained

$$f_r(X_1^0 + x_1, X_2^0 + x_2, \ldots, l_1^0 + \Delta l_1 + v_1, l_2^0 + \Delta l_2 + v_2, \ldots, d_1, \ldots, d_p) = 0. \tag{2}$$

Approximate values are introduced for the assessments and measured quantities. The expansion of (2) in Taylor series, limited to first-degree terms, after respective designations of the approximate values, partial derivatives and free terms, leads to

$$\begin{array}{l} t_{11}v_1 + t_{12}v_2 + \cdots + t_{1n}v_n + a_1x + b_1y + \cdots + \omega_1 = 0, \\ t_{21}v_1 + t_{22}v_2 + \cdots + t_{2n}v_n + a_2x + b_2y + \cdots + \omega_2 = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ t_{r1}v_1 + t_{r2}v_2 + \cdots + t_{rn}v_n + a_rx + b_ry + \cdots + \omega_r = 0, \end{array} \tag{3}$$

Respectively, system (3) may be written in the form

$$(4)\qquad \begin{aligned} &\underset{rn}{\mathbf{A}^*}\underset{n1}{\mathbf{v}} + \underset{rm}{\mathbf{B}}\underset{m1}{\mathbf{x}} + \underset{r1}{\boldsymbol{\omega}} = \underset{r1}{\mathbf{0}},\\ &\underset{km}{\mathbf{G}^*}\underset{m1}{\mathbf{x}} + \underset{k1}{\mathbf{d}} = \underset{k1}{\mathbf{0}}, \end{aligned}$$

where

$$(5)\qquad \underset{r1}{\boldsymbol{\omega}} = \underset{rn}{\mathbf{A}^*}\underset{n1}{\mathbf{l}} + \underset{r1}{\mathbf{c}}, \quad \boldsymbol{A}^* = \begin{bmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{r1} & t_{r2} & \dots & t_{rn} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} a_1 & b_1 & \dots & e_1 \\ a_2 & b_2 & \dots & e_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_r & b_r & \dots & e_r \end{bmatrix},$$
$$n > r + k > 0, \quad k < m, \quad r > m.$$

Here $\mathbf{B}$ is the so-called configuration or model matrix, containing the geometric relationships between the measured and unknown values.

Other k relationships are added in (4) to the relationships in (3), reflecting additional conditions between the unknowns. For example, in geodetic networks these may be conditions for the stability of primary elements: initial (baseline) coordinates, base, direction angle, thus eliminating the singularity of solution.

In this way there are already prerequisites and possibilities to actually derive the general case of adjustment according to MLS, which has been the objective of many authors in the 60s of the 20th century, for example Hristov [7].

2.2. General case of adjustment of correlated observations. The condition for a minimum in adjusting measurements according to MLS is

$$(6)\qquad \mathbf{v}^*\mathbf{Q}_l^{-1}\mathbf{v} = \min.$$

This is substantiated by the following prerequisites. During the adjustment the number of the measured quantities exceeds the unknown quantities and the conditions. Approximate values of the adjusted quantities can be calculated by the given ones on the basis of various combinations. Thus disclosures are obtained and it is not possible the predetermined conditions to be fulfilled. These are the functional relationships, for example the misclosures to be zero in the loops, the difference between summed elevation differences and the elevation differences calculated from the known elevations to be zero, etc. This means the measured quantities $\mathbf{l}$ should be corrected with corrections $\mathbf{v}$ in order the misclosures to be removed. The solution of Legendre and Gauss, obtained on the basis of condition (6), is the simplest one of all possible solutions for determining the corrections and obtaining unambiguous solution.

It is also known that when seeking a minimum for the available additional conditions, the function of Lagrange should be used (Joseph-Louis Lagrange, 1736–1813), which includes coefficients k_i, called correlates in geodesy.

The expanded type of the function of Lagrange in accordance with the prerequisites of Section 2.1, proposed by the author in 1973 [8], providing the most general case, i.e. adjustment of conditional equations with unknowns with additional conditions between these unknowns, taking into account [9], will have the form:

$$\underset{11}{\Phi} = \underset{1n}{\mathbf{v}^*}\underset{nn}{\mathbf{Q}_l^{-1}}\underset{n1}{\mathbf{v}} - 2\underset{1r}{\mathbf{k}_1^*}\left[\underset{rn}{\mathbf{A}^*}\underset{n1}{\mathbf{v}} + \underset{rm}{\mathbf{B}}\underset{m1}{\mathbf{x}} + \underset{r1}{\boldsymbol{\omega}}\right] - 2\underset{1k}{\mathbf{k}_2^*}\left[\underset{km}{\mathbf{G}^*}\underset{m1}{\mathbf{x}} + \underset{k1}{\mathbf{d}}\right]. \tag{7}$$

The proposal of the author consists in the introduction of another third term in the function of Lagrange $-2\underset{1k}{\mathbf{k}_2^*}\left[\underset{km}{\mathbf{G}^*}\underset{m1}{\mathbf{x}} + \underset{k1}{\mathbf{d}}\right]$, reflecting the existence of additional conditions between the unknowns and the given values, along with another vector with correlates, which is new, and the solution is given in [8]. Further on, except in [8], the solution is also shown in [9,10] and slightly modified in [11].

For the most general case of adjustment the basic system for determining the unknowns is obtained from the first derivatives of the function of Lagrange (7), set equal to zero, and after relevant transformations:

$$\begin{aligned} \mathbf{A}^*\mathbf{Q}_l\mathbf{A}\mathbf{k}_1 + \qquad\qquad\qquad\qquad &+ \mathbf{A}^*\mathbf{l} = \mathbf{0}, \\ \mathbf{B}^*\mathbf{k}_1 + \mathbf{G}\mathbf{k}_2 \qquad &= \mathbf{0}, \\ \mathbf{G}^*\mathbf{x} \qquad\qquad\qquad &+ \quad \mathbf{d} = \mathbf{0}. \end{aligned} \tag{8}$$

The system (8) is the basic one, from which after consecutive exclusion the correlates, the corrections and the unknowns are identified [8–10]:

- **Unknowns**

$$\mathbf{x} = \mathbf{N}_2^{-1}\left(\mathbf{B}^*\mathbf{N}_1^{-1}\mathbf{A}^* + \mathbf{G}\mathbf{N}_2^{-1}\mathbf{G}^*\mathbf{N}_2^{-1}\mathbf{B}^*\mathbf{N}_1^{-1}\mathbf{A}^*\right)\mathbf{l} - \mathbf{N}_2^{-1}\mathbf{G}\mathbf{N}_3^{-1}\mathbf{d} \tag{9}$$

- **Correlates**

$$\begin{aligned} \mathbf{k}_1 &= \left[\mathbf{N}_1^{-1}\mathbf{B}\mathbf{N}_2^{-1}\left(\mathbf{B}^*\mathbf{N}_1^{-1}\mathbf{A}^* - \mathbf{G}\mathbf{N}_3^{-1}\mathbf{G}^*\mathbf{N}_2^{-1}\mathbf{B}^*\mathbf{N}_1^{-1}\mathbf{A}^*\right) + \mathbf{N}_1^{-1}\mathbf{A}^*\right]\mathbf{l} \\ &\qquad + \mathbf{N}_1^{-1}\mathbf{B}\mathbf{N}_2^{-1}\mathbf{G}\mathbf{N}_3^{-1}\mathbf{d}, \\ \mathbf{k}_2 &= \mathbf{N}_3^{-1}\mathbf{G}^*\mathbf{N}_2^{-1}\mathbf{B}^*\mathbf{N}_1^{-1}\mathbf{A}^*\mathbf{l} - \mathbf{N}_3^{-1}\mathbf{d}. \end{aligned} \tag{10}$$

- **Corrections**

$$\begin{aligned} \mathbf{v} &= \mathbf{Q}_l\mathbf{A}\mathbf{N}_1^{-1}\left(\mathbf{B}\mathbf{N}_2^{-1}\mathbf{B}\mathbf{N}_1^{-1}\mathbf{A}^* - \mathbf{B}\mathbf{N}_2^{-1}\mathbf{G}\mathbf{N}_3^{-1}\mathbf{G}^*\mathbf{N}_2^{-1}\mathbf{B}^*\mathbf{N}_1^{-1}\mathbf{A}^* + \mathbf{A}^*\right)\mathbf{l} \\ &\qquad + \mathbf{Q}_l\mathbf{A}\mathbf{N}_1^{-1}\mathbf{B}\mathbf{N}_2^{-1}\mathbf{G}\mathbf{N}_3^{-1}\mathbf{d}. \end{aligned} \tag{11}$$

The following designations have been accepted:

$$\underset{rr}{\mathbf{N}_1} = \mathbf{A}^*\mathbf{Q}_l\mathbf{A}, \quad \underset{mm}{\mathbf{N}_2} = \mathbf{B}^*\mathbf{N}_1^{-1}\mathbf{B}, \quad \underset{kk}{\mathbf{N}_3} = \mathbf{G}^*\mathbf{N}_2^{-1}\mathbf{G}. \tag{12}$$

The following is valid for a function with adjusted values and its inverse weight

$$F_1 = F_1\left(l_1 + v_1, l_2 + v_2, \ldots, l_n + v_n, x_1, x_2, \ldots, x_m\right), \tag{13}$$

$$\frac{1}{P_F} = \begin{bmatrix} f_{1i}^* & f_{2i}^* \end{bmatrix} \begin{bmatrix} \mathbf{Q}_l - \mathbf{Q}_v & \mathbf{Q}_{l,x} \\ \mathbf{Q}_{x,l} & \mathbf{Q}_{xx} \end{bmatrix} \begin{bmatrix} f_{1i} \\ f_{2i} \end{bmatrix}. \tag{14}$$

Respectively, the mean error of a unit weight μ, and the mean errors of the unknowns m_{x_i} (these are σ standards in statistics and mean square errors m in geodesy) are:

$$\mu = \sqrt{\frac{1}{r+k-m}\mathbf{v}^*\mathbf{Q}_l^{-1}\mathbf{v}}, \tag{15}$$
$$m_{x_i} = \mu\sqrt{q_{ii}} \quad (q_{ii} \text{ are the diagonal elements of the matrix } \mathbf{Q}_x).$$

2.3. Specific cases of adjustment. The specific cases of adjustment of observations can be obtained on the basis of the general case. To this end the respective matrices not participating in a given specific case of adjustment are nullified [8–10]. So for adjustment of conditional observations with unknowns it is set $\mathbf{G} = \mathbf{0}$; for conditional observations $\mathbf{B} = \mathbf{0}$, $\mathbf{G} = \mathbf{0}$, for indirect observations with conditions between the unknowns $r = n$, $\mathbf{A}^* = -\mathbf{E}$, $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{0}$, for indirect observations $r = n$, $\mathbf{A} = -\mathbf{E}$, $\mathbf{G}^* = \mathbf{0}$, $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{0}$, for direct observations $r = n$, $m = 1$, $\mathbf{A} = -\mathbf{E}$, $\mathbf{G}^* = \mathbf{0}$, $\mathbf{B} = [1, 1, \ldots, 1]$.

3. Limits of validity of the linear model of MLS. The linear MLS model is widely used for solving geodetic problems. However, it is important to know the limits of its validity and when its correctness is violated a non-linear model should be used. To this end the magnitude and validity of the residual term in the expansion of function (2) in Taylor series, defining the functional model of adjustment, is numerically studied. Moreover, the study is model and numerical for the most frequently used case – adjustment of indirect observations, applied to adjust an angular-longitudinal three-dimensional network, built for investigating displacements and deformations [12].

The direction angles α_{ik}, slope distances D_{ik} and zenith angles Z_{ik}, defining the network, are included as measured quantities. In addition, the prerequisites necessary for expansion in Taylor series are valid for (2). In general form the residual term R_n [13] is:

$$R_n = \frac{F^{(n+1)}(\theta)}{(n+1)!}(X - X^0)^{(n-1)}, \tag{16}$$

where $F(X)$ is the function (2), X is the adjusted value, X^0 is the approximate value, θ is the number between X and X^0, n is the order of the derivative to which the Taylor expansion is restricted.

To establish whether the value of the residual term R_n is significant for the adjusted value X, it is important to compare its absolute value with the given (accepted) value – accuracy criterion ε. If

$$|R_n| < \varepsilon, \tag{17}$$

it is accepted that R_n does not influence the preliminarily selected model, and if

$$|R_n| > \varepsilon, \tag{18}$$

then the influence of R_n is significant and the selected model should be adjusted to include terms of higher degree from the Taylor expansion in series of the function $F(X)$.

In geodesy, actually in geodetic networks, mostly the influence of the second terms is of practical interest and for this reason the study is performed with restriction only to the second-degree terms. Then the residual term R_n is R_2 and the condition that the linear model is correct is

$$|R_2| < m, \tag{19}$$

where the mean error m_1 of the respective type of quantity l (α_{ik}, D_{ik}, Z_{ik}) is accepted as ε. If condition (19) is not satisfied the second-degree terms of the Taylor expansion of the respective function should be included, i.e. a non-linear model should be used.

Based on the known relationships

$$\begin{aligned}
\alpha_{ik} &= \arctan\frac{y_k - y_i}{x_k - x_i} = \arctan\frac{\Delta y_{ik}}{\Delta x_{ik}},\\
D_{ik} &= \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2 + (z_k - z_i)^2},\\
z_{ik} &= \operatorname{arccot}\frac{z_k - z_i}{\sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}}
\end{aligned} \tag{20}$$

the formulas of the residual terms are derived [12].

The calculation of the absolute values of the residual terms and their limitations is conducted for all combinations of the specified parameter values. It turned out to be rather complicated to summarize the results of the study for the individual types of investigated quantities. They are presented in an analytical (tables) and graphical form, automatically composed and plotted by specially developed software.

An example for the range of application of the linear model of the residual term for the slope distances $|R_2^{\delta D_{ik}}|$ is shown in the plot in Fig. 1.

The use of the linear or non-linear model depends on whether the restricting line is above or under the respective curve of the residual term. Its position

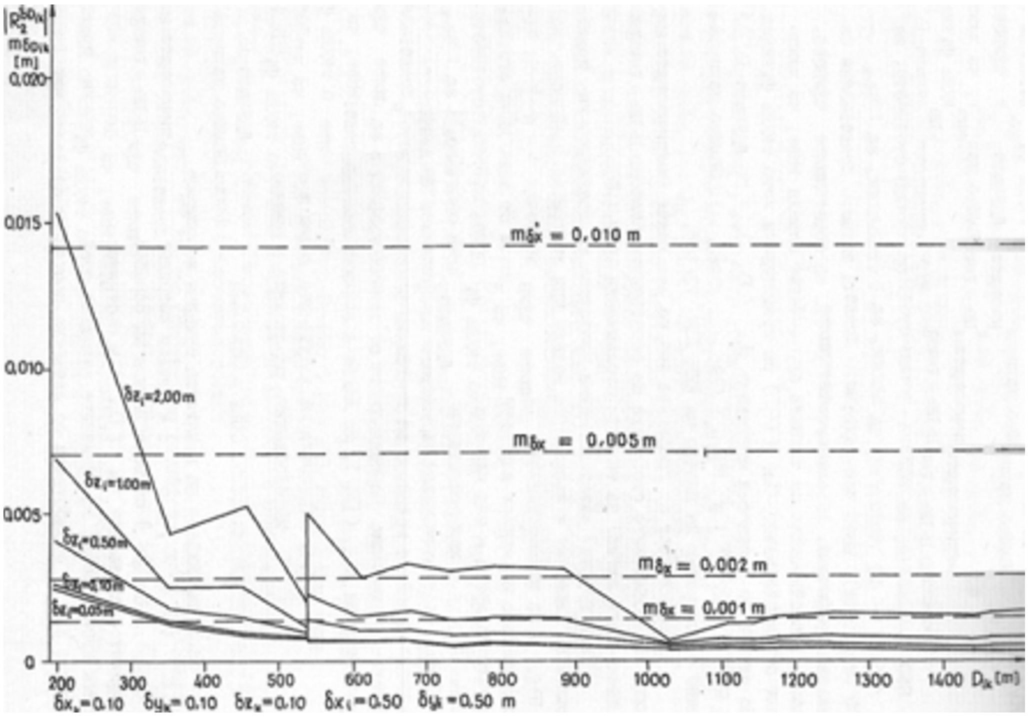

Fig. 1. Absolute values of a residual term for D and its limits

above the curve of the residual term means that for distances corresponding to the intersection of both lines the linear model is valid and for larger distances the non-linear model should be used.

The following more important conclusions can be drawn from the analysis of the summarized results:

1. The dependencies between d^r_{ik} (D_{ik}) and δx_{cp_r} for the functions of the direction angles, slope distances and zenith angles have the same character.

2. The established correct model (linear or non-linear) for adjustment of horizontal directions is also a correct model for adjustment of slope distances.

3. The influence of the second-degree terms in the Taylor series expansion on the zenith angle functions is greater than that on the direction angles and slope distances.

The model investigations actually comprise the possible case studies for determining the displacements of the points of the considered object, though realized under certain prerequisites. The determined generalized limit distances between the points and mean coordinate displacements allow verifying the linear model validity in each particular case.

It is established that the need for a non-linear model is not so rare and therefore it should not be neglected. In all cases the relevant evaluation should be made when there is doubt of this necessity, using the results of the conducted measurements and verifying the correctness of the linear model.

4. Comparison of Lagrangian functions in mechanics and geodesy and MLS application to solve variational problems in mechanics. The solution and formulas shown in Section 2 for the general and particular cases of adjustment represent a good and convenient prerequisite to implement these solutions in mechanics. This is appropriate and is based on the fact that the theory and practice of adjustment in geodesy are systematized, reasoned, well and consistently developed by geodesists, while in mechanics the issues are not so systematized and generalized. The application goes through the comparison of the functions of Laplace in geodesy and mechanics. Both functionals are given accordingly by equation (7) in geodesy as considered above and (equation (4.2.14) from [4,16]) or

$$(21)\qquad \begin{aligned} &\underset{11}{\Phi} = \underset{1n}{\mathbf{v}^*}\underset{nn}{\mathbf{Q}_l^{-1}}\underset{n1}{\mathbf{v}} - 2\underset{1r}{\mathbf{k}_1^*}\left[\underset{rn}{\mathbf{A}^*}\underset{n1}{\mathbf{v}} + \underset{rm}{\mathbf{B}}\underset{m1}{\mathbf{x}} + \underset{r1}{\boldsymbol{\omega}}\right] - 2\underset{1k}{\mathbf{k}_2^*}\left[\underset{km}{\mathbf{G}^*}\underset{m1}{\mathbf{x}} + \underset{k1}{\mathbf{d}}\right], \\ &\Pi_1 = \Pi + \int_{\mathbf{B}} \boldsymbol{\lambda}_{\boldsymbol{\varepsilon}}^*(\boldsymbol{\varepsilon} - \mathbf{BU})\, d\,\mathbf{B} + \int_{\mathbf{R}_u} \boldsymbol{\lambda}_u^*(\mathbf{U}_a - \mathbf{U}^p)\, d\mathbf{R}. \end{aligned}$$

In fact the comparison of their appearance indicates conformity, allowing also the comparison of their essence. Π corresponds to Φ and the requirement $\mathbf{v}^*\mathbf{Q}^{-1}\mathbf{v}$ for a minimum is valid for it. The expression $\boldsymbol{\varepsilon}-\mathbf{BU}$, which represents the description of the geometric compatibility, corresponds to the relationship $\mathbf{A}^*\mathbf{v}+\mathbf{Bx}+\boldsymbol{\omega}$. The conditions for displacement $\mathbf{U}_a - \mathbf{U}^p$ comply with the additional conditions $\mathbf{Gx}+\mathbf{d}$. The vectors $\boldsymbol{\lambda}_1$, respectively $\boldsymbol{\lambda}_2$, of the Lagrangian multipliers correspond to the correlates $\mathbf{k}_1$ and $\mathbf{k}_2$, compensating in mechanics the imperfection of the model and including the forces and stresses. The displacements and hence the deformations $\boldsymbol{\varepsilon}$ vary and acquire relevant corrections $\mathbf{v}$. Optimal approximation is achieved for the unknown displacements $\mathbf{U}_a$. They are replaced in the vector of unknowns $\mathbf{x}$, and together with the preset $\mathbf{U}^p$, compile the set of conditions for displacement and correspond to the quantities $\mathbf{d}$.

The accuracy estimation of the sought mechanical quantities is determined by adjustment according to direct observations (Section 2.3).

The determination of the variational unknowns in mechanics by MLS goes through the discretization of the medium, i.e. its division in small elements (up to triangles) according to certain principles so that it resembles a geodetic network, covering a section of a given object, for example a territory. To this end a regular grid is generated parallel to the coordinate axes, adhering to certain conditions, and as a result intersection – node points are formed. Since it is started with n elements with m node points, there is a redundancy, i.e. more elements than the necessary ones for unambiguous identification, which implies the possibility of MLS solution. Based on the already established analogy between the sought quantities, the different formulas from Section 2.2 can be used in the two functionals. In this way the determination of the variational unknowns is realized in general form. In this case this means that $\boldsymbol{\varepsilon}$ will be determined using formula (11), $\mathbf{U}_a$ – (9), $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ – (10). The accuracy estimates of the determined functions and unknowns can be obtained using formulas (14) and (15).

The weight problem in these solutions in mechanics is resolved taking under consideration the area of the triangles, length of ropes and type of the medium [11]. In the same manner the forces are derived from the displacements – the inverse problem of the one solved here, related to forces – displacements. In this context the relevant solution is made depending on the given elements.

5. Evaluation of the correctness of the model for geodetic network adjustment according to MLS. To estimate the accuracy, for example of the position of points in geodetic networks and the geometry of the networks, the accuracy along the coordinate axes, respectively the position of points is used. Often the criterion applied is the error ellipsis for two-dimensional networks and the error ellipsoid – for three-dimensional networks. They are used further on to optimize the networks. It should be clearly pointed out, however, that these criteria are very closely related to the choice of the origin and orientation of the coordinate systems. The magnitude of errors, respectively of the ellipses and ellipsoids, are functions of this choice. Moreover, the so-called law of error transfer in geodesy also applies here. But to what extent does it reflect the reality correctly and what is its effect on adjustment using MLS?

The accuracy of measurement of geodetic networks in their different parts is practically the same. This is due to the fact that the measurements are conducted using the same instruments, methods, operators, and under normal, not extreme atmospheric conditions. It should be expected that the equally accurate measured quantities after the MLS adjustment should yield equally accurate estimates of the adjusted quantities, for example coordinates and mean errors (standards), and they should be minimal and equal – homogeneous throughout the network. This means that the adjustment should not produce deformed results due to the model of adjustment. However, this is not so, as illustrated by the used network for investigating the landslide processes in the area of the town of Balchik (Fig. 2, [15,16]). Part of it (points 1-4) is located on the stable Dobrudzha plateau, and the other part – in the landslide zone. Point 1 is accepted as the origin of the local coordinate system and the $+y$ axis coincides with the side 1–2 of the network. It is seen in Fig. 1 that the error ellipses in the point coordinates increase with the distance from the origin of the coordinate system.

Undoubtedly attempts have been made to avoid this model and solutions have been proposed. A similar proposition is the so-called free adjustment of networks [17]. A whole theory, solutions and algorithms have been developed. So the origin of the coordinate system is practically located in the gravity centre

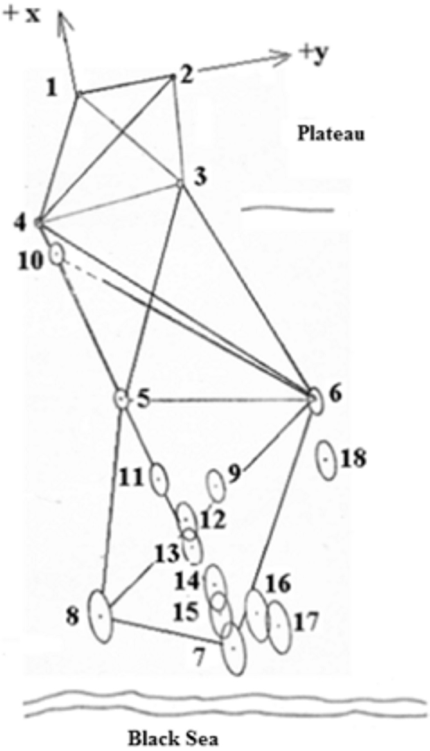

Fig. 2. Increasing of the error ellipses with the distances from the origin of the coordinate system

of the approximate coordinates of its points instead of in one of the ends of the network. It is true that in this way the distance from the origin to the most remote determinable point is decreased compared to the case, when the origin is for example in one of the ends of the network, and that the accuracy of the solution is improved. The problem, however, is not solved in this manner. Other additional "cosmetic" solutions are sought as optimization of the networks to decrease-optimize the size of the error ellipses (ellipsoids). So a kind of "juggling" is largely achieved without removing the root cause.

It follows from here that another adjustment model is necessary, so that the strong influence of the origin and orientation of the coordinate system is avoided and minimized and homogenized estimates are obtained for coordinate accuracy (mean errors, standards) and positions of points for further use in geodetic networks. This is especially important, for example, when using the mean errors as an element of the criteria for analysis and identification of the actually occurring displacements of points in the networks, built and repeatedly measured to determine and study the deformations of buildings, facilities and terrains – landslides.

Table 1

Coordinates, initial bases and mean square errors

No	Point	X/Y	Base/m_x/m_y (mm)										
	No	m	1–2	1–4	3–4	3–6	4–5	5–6	5–8	5–9	6–8	7–8	7–9
1	1	2000.000	.0	.0	1.5	2.3	2.5	4.8	6.1	5.7	5.5	8.3	7.6
2	2	2000.000	.0	1.4	2.0	2.4	2.7	4.6	6.0	5.6	5.5	8.3	7.5
3	3	1857.307	2.2	1.8	.0	.0	1.9	3.5	4.6	4.3	4.3	7.0	6.1
4	4	1840.516	3.0	.0	.0	2.0	.0	3.9	4.7	4.5	4.2	6.8	6.1
5	5	1587.694	6.9	4.3	3.5	2.5	.0	.0	.0	.0	1.9	3.6	2.4
6	6	1537.841	7.8	5.1	4.5	.0	2.4	.0	2.6	1.9	.0	3.7	2.1
7	7	1233.482	14.4	10.5	8.9	4.7	7.5	3.6	1.8	3.6	1.5	.0	.0
8	8	1307.130	12.9	9.3	7.6	3.7	6.3	3.4	.0	2.8	.0	.0	1.9
9	9	1449.618	9.4	6.2	5.1	1.6	2.9	1.2	2.0	.0	1.4	2.5	.0
10	1	1000.000	.0	.0	1.7	1.8	2.3	3.5	3.9	3.6	3.9	6.5	4.8
11	2	1129.140	.0	1.3	1.4	1.4	2.8	3.6	4.1	3.8	3.8	6.6	4.8
12	3	1111.930	1.5	1.5	.0	.0	2.1	2.5	3.0	2.7	2.8	5.2	3.6
13	4	914.166	2.5	.0	.0	2.1	.0	2.3	2.5	2.5	3.6	5.2	3.9
14	5	980.109	4.0	2.6	2.5	2.3	.0	.0	.0	.0	2.0	2.8	1.9
15	6	1234.778	6.2	5.4	3.8	.0	4.1	.0	2.6	2.1	.0	3.6	2.6
16	7	1068.649	7.7	5.9	5.2	3.9	3.6	3.0	2.5	2.2	2.1	.0	.0
17	8	897.660	7.8	5.7	5.8	5.4	3.3	3.0	.0	3.0	.0	.0	1.9
18	9	1083.303	5.4	4.2	3.3	2.3	2.5	1.6	1.3	.0	1.6	2.0	.0

Such a model for adjustment of precise geodetic networks (angular-longitudinal and height ones) and use in the overall system was proposed by the author together with relevant algorithms and software realization for the study of deformations [13, 14, 17, 18]. Only the part for network adjustment and mean error determination is presented here.

6. Adjustment of networks, determination of accuracy and further use of the solutions. The approximate coordinates are determined for the accepted initial elements and measured quantities and the network is adjusted (Fig. 1).

Pairs of points are consecutively accepted as initial ones, uniformly distributed in the single parts of the network, and adjustments are conducted with them. The respective point coordinates and their mean errors are obtained as a result for the single adjustments (Table 1) [14, 15].

It can be expected that the obtained coordinates of the points from adjustment with different initial bases are identical to those of the initial adjustment. This fact is true because the measurements used for all adjustments are the same and the pair of coordinates accepted as initial ones are the same as the adjusted ones during the initial adjustment.

T a b l e 2

Initial coordinates and mean square errors from the initial and consecutive adjustment

No	Point No	X/Y m	m_x/m_y (mm)	m_x/m_y min (mm)	m_x/m_y mean (mm)
1	1	2000.000	0	1.5	4.0
2	2	2000.000	0	1.4	4.2
3	3	1857.307	2.2	1.8	3.2
4	4	1840.516	3.0	2.0	3.2
5	5	1587.694	6.9	1.9	2.3
6	6	1537.841	7.8	1.9	2.7
7	7	1233.482	14.4	1.5	5.1
8	8	1307.130	13.0	1.9	4.4
9	9	1449.618	9.4	1.2	2.9
10	1	1000.000	0	1.7	2.9
11	2	1129.140	0	1.3	3.1
12	3	1111.930	1.5	1.5	2.3
13	4	914.166	2.5	2.1	2.2
14	5	980.109	4.0	1.9	1.7
15	6	1234.778	6.2	2.1	2.8
16	7	1068.649	7.7	2.1	3.3
17	8	897.660	7.8	1.9	3.3
18	9	1083.303	5.4	1.3	2.2

The mean square errors of the point coordinates, however, for the single adjustments are different depending on the initial origin, as seen in Table 1.

Representative values of the mean square errors for the individual points of the network (given in a row in Table 1) can be the minimal ones, other than zero values or mean arithmetic values for the line.

The initial coordinates and mean square errors for the adjustment and the respective minimal and mean arithmetic errors from the consecutive network adjustment are given in Table 2 on the basis of Table 1.

It is seen in Table 2 that the mean square errors of the points are homogeneous, with relatively equal size and minimal with the adjustment method proposed herein. This also applies to both errors – minimal and mean arithmetic, and they should be used in the further considerations and implementations of the networks. This is especially important in the study of deformations.

Thus a major drawback in the processing of geodetic network measurements according to MLS is avoided, when the adjustment model yields not real, higher and distorted estimates of the mean errors of the measured and adjusted quantities.

Based on this method of adjusting and producing true, more accurate, homogenized and minimal estimates for the adjusted quantities, a more accurate and effective application of MLS is achieved. This is especially important in the deformation studies of buildings, facilities, terrains and landslides. Here the minimized and homogenized standards (mean errors) are used in the criteria for evaluating stability and recording deformations. This increases the sensitivity of the criteria. Thus by the proposed algorithms and system of programs comparison and consistent statistical analysis are made in fact for the differences in the adjusted after certain measurement periods homonymous invariant elements (angles and lengths – corresponding to the measured ones or specially preset), as well as of the point coordinates in the independent two-dimensional angular, longitudinal or angular-longitudinal geodetic networks [19]. Moreover, as already mentioned, the stability of the initial and the value of the actual displacements (coordinate and vector) of the other points of the network are established. The procedure of verifying the statistical hypotheses and Student's distribution are used [8–10, 13–15, 17].

7. Conclusion. The systematization, development, summary and evaluation of the correctness and applicability of MLS conducted herein, together with its improvement and implementation for the purposes of geodesy and mechanics, including for particular applications in investigating deformations, gives contemporary geodetic assessment of the method of least squares, which should also be a prerequisite for MLS further development and application, especially in geodesy and mechanics.

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